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ANOMALOUS KINETICS OF HARD CHARGED PARTICLES: DYNAMICAL RENORMALIZATION GROUP RESUMMATION

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Abstract

We study the kinetics of the distribution function for charged particles of hard momentum in scalar QED. The goal is to understand the effects of infrared divergences associated with the exchange of quasistatic magnetic photons in the relaxation of the distribution function. We begin by obtaining a kinetic transport equation for the distribution function for hard charged scalars in a perturbative expansion that includes hard thermal loop resummation. Solving this transport equation, the infrared divergences arising from absorption and emission of soft quasi-static magnetic photons are manifest in logarithmic secular terms. We then implement the dynamical renormalization group resummation of these secular terms in the relaxation time approximation. The distribution function (in the linearized regime) is found to approach equilibrium as $\delta n_k(t) = \delta n_k(t_o)\; e^{-2\alpha T (t-t_o)\ln[(t-t_o)\bar{\mu}]}$, with $\bar{\mu} \approx \omega_p$ the plasma frequency and $\alpha = e^2/4\pi$. This anomalous relaxation is recognized to be the *square* of the relaxation of the single particle propagator, providing a generalization of the usual relation between the damping and the interaction rate. The renormalization group approach to kinetics reveals clearly the time scale $t_{rel} \approx (\alpha\, T \ln[1/\alpha])^{-1}$ arising from infrared physics and hinges upon the separation of scales $t_{rel} >> \omega_p^{-1}$.

I. INTRODUCTION

The possibility of probing the quark-pluon plasma at the forthcoming RHIC at BNL and LHC at CERN, has sparked an intense activity in the understanding of collective excitations in ultrarelativistic plasmas and their potential experimental signatures (for reviews see [1]-

[10]). An important issue in this program is a consistent assessment of the time scales for thermalization and equilibration of the quasiparticles and their distribution functions, i.e. the damping and interaction rates [9–13]. At high temperatures usual perturbation theory breaks down, but a consistent resummation program developed by Braaten and Pisarski [14]-[18] provides a systematically improved perturbative expansion known as the hard thermal loop resummation (HTL). The HTL resummation incorporates screening corrections in a gauge invariant manner and is sufficient to render finite the damping rate of excitations at rest in the plasma [19] and transport cross sections [20].

However these corrections are not sufficient to cure the infrared sensitivity of the damping rate of hard charged excitations which is dominated by the exchange of quasistatic magnetic photons and or gluons [21,22]. Whereas in QCD it is expected that a non-perturbative magnetic mass will provide an infrared cutoff and ameliorate the infrared sensitivity of the damping rate for hard charged particles [21,22], in QED the transverse photons do not acquire a magnetic mass, screening is only dynamical through Landau damping and therefore the infrared singularities remain, possibly to all orders. This infrared sensitivity in the case of QED has led some authors to question the validity of the quasiparticle description and exponential relaxation that is the main concept behind the calculation of the damping rate [23,24].

These questions were recently clarified by the implementation of a Bloch-Nordsieck (eikonal) resummation of the infrared divergent diagrams in QED [25,26] which leads to an anomalous real-time relaxation of the electron propagator of the form $\approx \exp[-\alpha\, T\, t \ln(t\, \omega_p)]$. This result has been recently confirmed in scalar QED (SQED) by implementing a dynamical renormalization group resummation directly in real time [27]. The advantage of this dynamical renormalization group approach is that it provides a real-time description of relaxation bypassing the limitation of the quasiparticle interpretation. In reference [27] the equivalence of the dynamical renormalization group to well studied situations at zero temperature was established: for cases with infrared threshold singularities at zero temperature it is equivalent to the Bloch-Nordsieck and the usual Euclidean renormalization group resummation. In the absence of infrared threshold singularities it leads to the real time evolution obtained by more conventional methods (for a more complete comparison the reader is referred to [27]). For SQED at finite temperature it leads to the relaxation of the single particle propagator which agrees exactly with the results of the eikonal (Bloch-Nordsieck) approximation [25] in QED. This is one more example of the similarities between QED, QCD and SQED [28] in lowest order in the HTL resummation.

In this article we explore a different but related question: since in hot QED and in SQED the infrared divergences associated with the emission and absorption of quasi-static magnetic photons lead to anomalous, non-exponential relaxation of the propagator of charged excitations [25–27], what is the kinetic equation that describes the relaxation of the *distribution function* for these excitations?. This kinetic equation would be the equivalent of the Boltzmann equation for the distribution function.

When the quasiparticle picture is valid, the relaxation time approximation leads to the linearized equation $\delta\dot{N}_k(t) = -\gamma_k\ \delta N_k(t)$ with $\delta N_k(t)$ is the departure from equilibrium of

the quasiparticle distribution function and the relaxation rate γ_k (inverse of the relaxation time) is time independent and simply related to the damping rate for single quasiparticles Γ_k: $\gamma_k = 2\,\Gamma_k$ [9–11]. If the single quasiparticle propagator is *not* exponentially damped it is reasonable to expect that the linearized kinetic equation for the distribution function will require a time dependent generalization of γ_k.

The goals of this article are a): a derivation of the proper kinetic equation for the case of hard charged (quasi) particles, b) interpretation of the relaxation time approximation and comparison to the relaxation of the single (quasi) particle Green's function.

We study these issues within the context of SQED to make contact with previous results [27]. This model *is* relevant to study the physics of the QGP because to lowest order in α SQED has the same infrared divergence and HTL structure as both QED and QCD [28].

The strategy: Our strategy leading to the kinetic equation follows closely the derivation of the Boltzmann equation presented in [29,30]. It begins by defining a suitable number operator N_k. In the case under consideration that of a hard scalar with momentum $k \geq T$ the collective mode and the (renormalized) particle are indistinguishable [28,10], thus the number operator is the usual operator associated with asymptotic states. The time derivative of this operator is obtained from the Heisenberg equations of motion, and its average over an initial density matrix is performed using perturbation theory in terms of the non-equilibrium propagators [29,30]. We find that the time evolution of N_k obtained in this perturbative expansion contains logarithmic *secular* terms in time. The dynamical renormalization group program is invoked to resum these secular terms. The resummed distribution function obeys a dynamical renormalization group equation which is recognized to be the generalization of the Boltzmann equation [31].

The results: There are two main results of our study, one of general scope and the other particular to the anomalous kinetics of hard charged particles in SQED (but likely to be shared by QED and QCD in lowest order in HTL resummation): a) we propose a microscopic derivation of quantum kinetic equations beginning from a non-equilibrium perturbative expansion in real time and using the dynamical renormalization group to resum the secular terms. This formulation allows to include other resummation schemes such as HTL directly in the derivation of the kinetic equation. b) More specifically to the problem of the kinetics of the distribution function for hard charged particles, we have focused on SQED as a model that bears many of the relevant features of QED and QCD. We find that the linearized version of the kinetic equation (relaxation time approximation) for this distribution function evolves in time as $\delta n_k(t) = \delta n_k(t_o)\ e^{-2\,\alpha T\,(t-t_o)\,\ln\bar{\mu}(t-t_o)}$ for $t >> t_o$ (with $\alpha = e^2/(4\pi)$ the 'fine structure constant' and $\bar{\mu} \approx \omega_p$). This must be compared to the real-time evolution of the hard scalar propagator $G_k(t) \approx e^{-\alpha\,T\,t\,\ln\bar{\mu}t}$ [27] which reveals a similar relation between the relaxation time scales of the single particle Green's function and that of the distribution function as in the usual quasiparticle picture but in a manner that is *not* associated with pure exponential relaxation.

A similar anomalous relaxation has been found by [25] in spinor QED via the Bloch-Nordsieck (eikonal) approximation.

The article is organized as follows: in section II we introduce the theory and describe

the perturbative framework that leads to the kinetic equation to $\mathcal{O}(\alpha)$. The different contributions are interpreted as in-medium processes. In section III we focus on the relaxation time approximation (linearized kinetic equation), recognize the secular terms arising in the perturbative expansion and implement the dynamical renormalization group resummation of these secular terms in the asymptotic long time limit. In section IV we provide an interpretation of the resummation scheme resulting from the dynamical renormalization group and generalize the solution to include short time transients. Section V summarizes our conclusions, provides an assessment of the potential impact of these results and poses other questions.

II. THE KINETIC EQUATION

We propose to study the relaxation of hard charged scalars in scalar QED as a prelude to studying the more technically involved cases of QED and QCD. Scalar QED shares many of the important features of QED and QCD in leading order in the HTL resummation [28]. Furthermore, the infrared physics in QED captured in the eikonal approximation (Bloch-Nordsieck) as clearly explained in [25] has been reproduced recently via the dynamical renormalization group in SQED [27], thus lending more support to the similarities of both theories at least in leading HTL order.

In the Abelian theory under consideration, it is rather straightforward to implement a gauge invariant formulation by projecting the Hilbert space on states annihilated by Gauss' law. Gauge invariant operators can be constructed and the Hamiltonian and Lagrangian can be written in terms of these. The resulting Lagrangian is exactly the same as that in Coulomb gauge [30] (for more details see [32]) and is given by

$$
\begin{aligned}
\mathcal{L} = {} & \partial_\mu \Phi^\dagger \, \partial^\mu \Phi - m^2 \Phi^\dagger \Phi + \frac{1}{2} \partial_\mu \vec{A}_T \cdot \partial^\mu \vec{A}_T - e \vec{A}_T \cdot \vec{j}_T - e^2 \vec{A}_T \cdot \vec{A}_T \ \Phi^\dagger \Phi + \\
& + \frac{1}{2} \left(\nabla A_0\right)^2 + e^2 A_0^2 \ \Phi^\dagger \Phi + e A_0 \ \rho \ , \\
\vec{j}_T = {} & i(\Phi^\dagger \vec{\nabla}_T \Phi - \vec{\nabla}_T \Phi^\dagger \Phi) \quad ; \quad \rho = -i \left(\Phi \dot{\Phi}^\dagger - \Phi^\dagger \dot{\Phi} \right) \ .
\end{aligned}
$$

where we have traded the instantaneous Coulomb interaction for a gauge invariant Lagrange multiplier field A_0 which should not be confused with a time component of the gauge field. $\vec{A}_T$ is the transverse component satisfying $\vec{\nabla} \cdot \vec{A}_T(\vec{x}, t) = 0$. Since we are only interested in obtaining the infrared behavior arising from finite temperature effects we do not introduce the renormalization counterterms to facilitate the study, although these can be systematically included in our formulation [27]. The finite temperature behavior is ultraviolet finite. The non-equilibrium generating functional requires the fields on the forward and backward time branches as discussed in [30] (for more details on the non-equilibrium formulation see [33]-[35]).

We anticipate that the Coulomb interaction will not be relevant to the infrared physics because the longitudinal photon will be screened with a screening mass $m_s \propto eT$. Only

the transverse photons will lead to infrared divergences [21,25,27] and therefore we neglect the contribution from longitudinal photons (the Coulomb interaction). Furthermore we will consider a neutral system with vanishing chemical potential. It is convenient to perform a spatial Fourier transform at some given time and write

$$\Phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \, \phi(\vec{k}) \, e^{i\vec{k}\cdot\vec{x}} \;\; ; \;\; \phi(\vec{k}) = \frac{1}{\sqrt{2\omega_k}} \left[a_k + b^{\dagger}_k\right]$$
$$\dot{\Phi}(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \, \dot{\phi}(\vec{k}) \, e^{i\vec{k}\cdot\vec{x}} \;\; ; \;\; \dot{\phi}(\vec{k}) = -i\sqrt{\frac{\omega_k}{2}} \left[a_k - b^{\dagger}_k\right]$$

And the spatial Fourier transform for the transverse gauge field

$$\vec{A}_T(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \, \vec{\mathcal{A}}_T(\vec{k}) \, e^{i\vec{k}\cdot\vec{x}} \quad .$$

Since we are interested in hard charged massive scalars, $\omega_k \approx k$ the collective modes are indistinguishable from the single particle excitations [28,10]. The soft collective modes require a definition of the quasiparticle number operator, for which the formulation recently proposed in terms of quasiparticles [36] could prove useful. This case will be explored elsewhere.

The number of positively charged scalars (which at zero chemical potential is equal to the number of negatively charged scalars) is then given by the following Heisenberg operator

$$\begin{aligned} n_+(k,t) = a^{\dagger}_k(t)a_k(t) = \frac{1}{2\omega_k} \Big[& \dot{\phi}^{\dagger}(\vec{k},t)\dot{\phi}(\vec{k},t) + \omega_k^2 \, \phi^{\dagger}(\vec{k},t)\phi(k,t) \\ & + i\omega_k \left(\phi^{\dagger}(\vec{k},t)\dot{\phi}(\vec{k},t) - \dot{\phi}^{\dagger}(\vec{k},t)\phi(\vec{k},t)\right)\Big] \end{aligned}$$

Using the Heisenberg equations of motion we obtain

$$\begin{aligned} \dot{n}_+(k,t) = & \frac{e}{\omega_k} \int \frac{d^3q}{(2\pi)^{3/2}} \vec{K}_T(\vec{q}) \cdot \vec{\mathcal{A}}_T(\vec{q},t)\phi^{\dagger}(\vec{k}+\vec{q},t) \left(\frac{\partial}{\partial t} - i\omega_k\right)\phi(\vec{k},t) \\ & + \frac{e}{\omega_k}\left[\left(\frac{\partial}{\partial t} + i\omega_k\right)\phi^{\dagger}(\vec{k},t)\right] \int \frac{d^3q}{(2\pi)^{3/2}} \vec{K}_T(\vec{q}) \cdot \vec{\mathcal{A}}_T(\vec{q},t)\phi(\vec{k}-\vec{q},t) \\ \vec{K}_T(q) = & \; \vec{k} - \hat{\vec{q}}\,(\vec{k}\cdot\hat{\vec{q}}) \end{aligned}$$

We need to take the expectation value of this Heisenberg operator in the initial density matrix. The following identities prove useful

$$\begin{aligned} Tr\left[A(t)B(t)C(t)\rho(t_i)\right] & = Tr\left[B(t)C(t)\rho(t_i)A(t)\right] = \langle B^+(t)C^+(t)A^-(t)\rangle \\ & = Tr\left[C(t)\rho(t_i)A(t)B(t)\right] = \langle C^+(t)A^-(t)B^-(t)\rangle \end{aligned}$$

where the $\pm$ correspond to operators defined on the forward $(+)$ and backward $(-)$ time branches, which are obtained as functional derivatives with respect to sources of the forward and backward time evolution operators [33]- [35]. These identities allow to extract the time derivatives outside of the expectation values and to avoid the potential Schwinger terms

associated with time ordering. Now the expectation value of $\dot{n}_+(k,t)$ given above can be written as follows

$$\langle \dot{n}_+(k,t)\rangle = \frac{e}{\omega_k}\left(\frac{\partial}{\partial t'} - i\omega_k\right)\int \frac{d^3q}{(2\pi)^{3/2}} K_{i,T}(q)\langle \mathcal{A}^-_{i,T}(\vec{q},t)\phi^{\dagger,-}(\vec{k}+\vec{q},t)\phi^+(\vec{k},t')\rangle\,|_{t=t'} + \frac{e}{\omega_k}\left(\frac{\partial}{\partial t'} + i\omega_k\right)\int \frac{d^3q}{(2\pi)^{3/2}} K_{i,T}(q)\langle \mathcal{A}^+_{i,T}(\vec{q},t)\phi^{+}(\vec{k}-\vec{q},t)\phi^{\dagger,-}(\vec{k},t')\rangle\,|_{t=t'}$$

This expectation value is computed in non-equilibrium perturbation theory in terms of the non-equilibrium propagators and vertices. Since we are interested in the infrared region in the internal loop momenta, we must consider the HTL resummed photon propagator, but the scalar propagator need not be resummed because $k \geq T$ and the scalar in the loop is hard and massive. The required non-equilibrium propagators are the following [30,27,35]
•Scalar Propagators (zero chemical potential)

$$\langle \phi^{(a)\dagger}(\vec{p},t)\phi^{(b)}(-\vec{p},t')\rangle = -iG^{ab}_p(t,t')$$

where $(a,b)\ \in \{+,-\}$.

$$\begin{aligned}
&G^{++}_p(t,t') = G^>_p(t,t')\Theta(t-t') + G^<_p(t,t')\Theta(t'-t)\ ,\\
&G^{--}_p(t,t') = G^>_p(t,t')\Theta(t'-t) + G^<_p(t,t')\Theta(t-t')\ ,\\
&G^{\pm\mp}_p(t,t') = G^{<(>)}_p(t,t')\ ,\\
&G^>_p(t,t') = \frac{i}{2\omega_p}\left[(1+n_p)\,e^{-i\omega_p(t-t')} + n_p\,e^{i\omega_p(t-t')}\right]\ ,\\
&G^<_p(t,t') = \frac{i}{2\omega_p}\left[n_p\,e^{-i\omega_p(t-t')} + (1+n_p)\,e^{i\omega_p(t-t')}\right]\ ,\\
&n_p = \frac{1}{e^{\beta\omega_p}-1}\ .
\end{aligned}$$

•HTL dressed Photon Propagators [10,27]

$$\langle \mathcal{A}^{(a)}_{Ti}(\vec{q},t)\mathcal{A}^{(b)}_{Tj}(-\vec{q},t')\rangle = -i\mathcal{G}^{ab}_{ij}(\vec{q};t,t')$$

$$\begin{aligned}
&\mathcal{G}^{++}_{ij}(q;t,t') = \mathcal{P}_{ij}(\vec{q})\ \left[\mathcal{G}_q{>}(t,t')\Theta(t-t') + \mathcal{G}^<_q(t,t')\Theta(t'-t)\right]\ ,\\
&\mathcal{G}^{--}_{ij}(q;t,t') = \mathcal{P}_{ij}(\vec{q})\ \left[\mathcal{G}^>_q(t,t')\Theta(t'-t) + \mathcal{G}^<_q(t,t')\Theta(t-t')\right]\ ,\\
&\mathcal{G}^{\pm\mp}_{ij}(q;t,t') = \mathcal{P}_{ij}(\vec{q})\ \mathcal{G}^{<(>)}_q(t,t')\ ,\\
&\mathcal{G}^>_q(t,t') = \frac{i}{2}\int dq_o\ \tilde{\rho}_T(q_o;\vec{q})\left[(1+N_{q_o})e^{-iq_o(t-t')} + N_{q_o}e^{iq_o(t-t')}\right]\ ,\\
&\mathcal{G}^<_q(t,t') = \frac{i}{2}\int dq_o\ \tilde{\rho}_T(q_o;\vec{q})\left[N_{q_o}\,e^{-iq_o(t-t')} + (1+N_{q_o})\,e^{iq_o(t-t')}\right],\\
&N_{q_o} = \frac{1}{e^{\beta q_o}-1}\ .
\end{aligned}$$

Where we have used the properties [27,10]

$$\tilde{\rho}_T(-q_o,q) = -\tilde{\rho}_T(q_o,q) \quad ; \quad N(-q_o) = -[1+N(q_o)] \tag{2.1}$$

The HTL spectral density is given by [27,28,30]

$$\tilde{\rho}_T(q_0,q) = \frac{1}{\pi}\frac{\Sigma_I(q_0,q)\;\Theta(q^2-q_0^2)}{\left[q_0^2-q^2-\Sigma_R(q_0,q)\right]^2+\Sigma_I^2(q_0,q)} + \mathrm{sign}(q_0)\;Z(q)\;\delta(q_0^2-\omega_p^2(q)) \tag{2.2}$$

$$\Sigma_I(q_0,q) = \frac{\pi e^2T^2}{12}\frac{q_0}{q}\left(1-\frac{q_0^2}{q^2}\right) \tag{2.3}$$

$$\Sigma_R(q_0,q) = \frac{e^2T^2}{12}\left[2\,\frac{q_0^2}{q^2}+\frac{q_0}{q}\left(1-\frac{q_0^2}{q^2}\right)\ln\left|\frac{q_0+q}{q_0-q}\right|\right]$$

where $\omega_p(q)$ is the plasmon pole and $Z(q)$ its (momentum dependent) residue, which will not be relevant for the following discussion. The important feature of this HTL resummed spectral density is its support below the light cone, i.e. for $q^2 > q_0^2$, the imaginary part (2.3) originates in the process of Landau damping [21] from scattering of quanta in the medium. Above $\mathcal{P}_{ij}(\vec{q})$ is the transverse projection operator:

$$\mathcal{P}_{ij}(\vec{q}) = \delta_{ij} - \frac{q_iq_j}{q^2}\;.$$

Using the above expressions for the non-equilibrium propagators, and after some tedious but straightforward algebra, we find the expectation value $\langle \dot{n}_k(t)\rangle$ to lowest order in perturbation theory $\mathcal{O}(e^2)$ is given by

$$\begin{aligned}\langle \dot{n}_k(t)\rangle = {} & \frac{e^2k^2}{4\pi^2\omega_k}\int_0^\infty \frac{q^2dq}{\omega_{\vec{k}+\vec{q}}}\int_{-1}^{1} d\cos\theta\;(1-\cos^2\theta)\int_{-\infty}^{\infty} dq_o\;\tilde{\rho}_T(q_o,q)\int_{t_i}^{t} dt' \times \\ & \Big\{\Big[(1+N_{q_o})(1+n_{\vec{k}+\vec{q}})(1+n_k) - N_{q_o}\,n_{\vec{k}+\vec{q}}\,n_k\Big]\cos(q_o+\omega_{\vec{k}+\vec{q}}+\omega_k)(t-t')+ \\ & \Big[N_{q_o}(1+n_{\vec{k}+\vec{q}})(1+n_k) - (1+N_{q_o})\,n_{\vec{k}+\vec{q}}\,n_k\Big]\cos(-q_o+\omega_{\vec{k}+\vec{q}}+\omega_k)(t-t')+ \\ & 2\Big[N_{q_o}\,n_{\vec{k}+\vec{q}}\,(1+n_k) - (1+N_{q_o})(1+n_{\vec{k}+\vec{q}})\,n_k\Big]\cos(q_o+\omega_{\vec{k}+\vec{q}}-\omega_k)(t-t')\Big\}\end{aligned} \tag{2.4}$$

where θ is the angle between $\vec{k}$ and $\vec{q}$. The different contributions have a very natural interpretation in terms of gain – loss processes. The first term in brackets corresponds to the process $0 \to \gamma^* + s + \bar{s}$ minus the process $\gamma^* + s + \bar{s} \to 0$, the second term corresponds to $\gamma^* \to s + \bar{s}$ minus $s + \bar{s} \to \gamma^*$, and the last term corresponds to the scattering in the medium $\gamma^* + s \to s$ minus the inverse process $s \to \gamma^* + s$ where γ^* refers to the HTL- dressed photon and s, $\bar{s}$ refer to the charged quanta of the scalar field Φ.

III. RELAXATION TIME APPROXIMATION: SECULAR TERMS

We will assume that there is an equilibrium solution for the distribution function and that at time $t = t_0$ the distribution function for a fixed mode k is disturbed slightly off

equilibrium so that $n_k(t<t_0)=n_k^{eq}$; $n_k(t=t_0)=n_k^{eq}+\delta n_k(t_0)$ while the rest of the modes remain in equilibrium. We want to study the time evolution of the perturbed distribution in the linearized approximation. This leads to a kinetic equation (2.4) that is linear in δn_k, since consistent with the condition that only the mode of wavevector k is off equilibrium, we set $\delta n_{\vec{k}+\vec{q}}=0$ for $\vec{q}\neq 0$ and the point $\vec{q}=0$ does not contribute because the integration measure vanishes there.

This is known as the relaxation time approximation. Only in this approximation the relaxation of the distribution function for the number of particles is related to the relaxation of single particle Green's function. Since the propagators entering in the perturbative expansion of the kinetic equation are in terms of the distribution functions at the initial time the time integration can be done straightforwardly leading to the following linearized equation

$$
\begin{aligned}
\delta \dot{n}_k(t) = \delta n_k(t_0)\, \frac{e^2k^2}{4\pi^2\omega_k} \int_0^\infty \frac{q^2dq}{\omega_{\vec{k}+\vec{q}}} \int_{-1}^{+1} (1-\cos^2\theta)\; d\cos\theta \; \int_{-\infty}^{+\infty} dq_o\; \tilde{\rho}_T(q_o,q)\, \times \\
\Bigg\{ 2\left[1+N_{q_o}+n_{\vec{k}+\vec{q}}\right] \frac{\sin(q_o+\omega_{\vec{k}+\vec{q}}+\omega_k)(t-t_0)}{q_o+\omega_{\vec{k}+\vec{q}}+\omega_k} \\
-\; 2\left[1+N_{q_o}+n_{\vec{k}+\vec{q}}\right] \frac{\sin(q_o+\omega_{\vec{k}+\vec{q}}-\omega_k)(t-t_0)}{q_o+\omega_{\vec{k}+\vec{q}}-\omega_k} \Bigg\}
\end{aligned} \tag{3.1}
$$

where we have used the properties (2.1) to combine the first and second terms in the kinetic equation (2.4).

Before proceeding further, it is illustrative to analyze the above equation in the long time limit $t>>t_o$. *If* the denominators in (3.1) have isolated zeroes in the region of integration we can use the approximation $\sin[\Omega\tau]/\Omega \overset{\tau\to\infty}{\approx} \pi\delta(\Omega)$ which is the usual approximation leading to Fermi's Golden rule. Thus *if* there are no singularities arising from the integrals and if the resulting delta functions are satisfied one finds a linear secular term in time upon integrating in time the rate equation (3.1). This is a perturbative signal of pure exponential relaxation at long times [31]. However in the case under consideration there are infrared singularities in the integrand [25,27] and the long-time limit must be studied carefully.

For this purpose it is convenient to introduce the following spectral density

$$
\begin{aligned}
\rho(k;\omega) = -\frac{e^2k^2}{\pi^2} \int_0^\infty \frac{q^2dq}{\omega_{\vec{k}+\vec{q}}} \int_{-1}^{1} (1-\cos^2\theta)\; d\cos\theta \; \int_{-\infty}^{\infty} dq_o\; \tilde{\rho}_T(q_o,q)\, \times \\
\left[1+N(q_o)+n_{\vec{k}+\vec{q}}\right]\; \delta(\omega-q_o-\omega_{\vec{k}+\vec{q}})
\end{aligned} \tag{3.2}
$$

which is the same as that studied within the context of the relaxation of the amplitude of a mean field in [27] and in the eikonal approximation [25].

In terms of this spectral density we obtain the time derivative of distribution function in the form

$$
\begin{aligned}
\delta\dot{n}_k(t) &= -\alpha\;\Gamma_k(t)\;\delta n_k(t_0) \\
\alpha\Gamma_k(t) &= -\frac{1}{2\omega_k}\int d\omega\;\rho(k;\omega)\left[\frac{\sin[(\omega-\omega_k)(t-t_o)]}{(\omega-\omega_k)}+(\omega\to-\omega)\right] \quad ; \quad \alpha=\frac{e^2}{4\pi}
\end{aligned} \tag{3.3}
$$

which upon integrating in time with initial condition at t_o leads to the form

$$\delta n_k(t) = \delta n_k(t_0) \left\{ 1 - \alpha \int_{t_o}^{t} \Gamma_k(t')\, dt' \right\}$$

We focus on the case of hard external momemtum $k \approx T$ but on the infrared region of the loop integral $q_o,\ q \leq eT$ in the spectral function (3.2) [25,27]. This is the region dominated by the exchange of soft, HTL resummed transverse photons [21,25] and that dominates the long time evolution of the distribution function. In this region we can replace $\omega_k \approx k$; $\omega_{\vec{k}+\vec{q}} \approx k + q \cos\theta$.

Potential secular terms (growing in time) could arise in the long time limit $t >> t_o$ whenever the denominators in (3.3) vanish, i.e. for the region of frequencies $\omega \approx \omega_k \approx k$ and $\omega \approx -\omega_k \approx -k$. For $\omega \approx \omega_k \approx k$ we see that the argument of the delta function in (3.2) vanishes in the region of the Landau damping cut of the exchanged transverse photon $q_o^2 < q^2$ and contributes to the infrared behavior. On the other hand, for $\omega \approx -\omega_k \approx -k$ the delta function in (3.2) is satisfied for $q_o \approx -2k$, and this region gives a negligible contribution to the long time dynamics. Therefore only the first term in (3.3) (with $\omega - \omega_k$) contributes in the long time limit. This term is dominated by the Landau damping region of the spectral density of the exchanged soft photon given by (2.2) since for $\omega \approx \omega_k$ the argument of the delta function is $q_o + q\ \cos(\theta)$ and this is the region where the imaginary part (2.3) has support. The second contribution (with $\omega + \omega_k$) oscillates in time and is always bound and perturbatively small.

In the hard limit $\omega_k \approx k$, and for $q_o; q \leq eT << T$ we neglect the contribution from $n_{\vec{k}+\vec{q}}$ (hard and massive scalar) and replace $N(q_o) \approx T/q_o$. The spectral density is found to be given by [27]

$$\rho(k;\omega) \overset{\omega \to k}{=} \frac{e^2 kT}{\pi^2} \ln \left| \frac{\omega - k}{\mu} \right| + \mathcal{O}(\omega - k)$$

where μ is an infrared cutoff $\mu \approx \omega_p \approx eT$ [21,25,27].

In the limit $\mu(t - t_o) >> 1$ we find [27]

$$\int_{t_o}^{t} \Gamma_k(t')dt' \overset{\mu(t-t_o)>>1}{=} 2T\,(t - t_o) \ln[\bar{\mu}(t - t_o)] + \text{non-secular terms} \tag{3.4}$$

with $\bar{\mu} = \mu \exp[\gamma_E - 1]$ and γ_E is the Euler-Mascheroni constant. In lowest order in perturbation theory, the occupation number that enters in the loops are those at the initial time. Obviously perturbation theory breaks down at time scales $t - t_o \approx |\alpha T \ln \alpha|^{-1}$ where we used $\bar{\mu} \approx \omega_p \approx eT$. This situation is similar to the case of a weakly damped harmonic oscillator when a perturbative solution in terms of the damping coefficient is sought. Such a perturbative solution has secular terms that grow linearly in time in lowest order, which reflect the perturbative expansion of the damping exponential, i.e. $e^{-\gamma t} \approx 1 - \gamma t + \cdots$. This perturbative expansion breaks down at time scales $\mathcal{O}(1/\gamma)$. The dynamical renormalization group is a systematic generalization of multi-time scale analysis and sums the secular terms, thus improving the perturbative expansion [37]. For a discussion of the dynamical renormalization group in other contexts, including applications to quantum field theory problems see [37]- [40].

We now implement the dynamical renormalization group to sum the secular divergences and to improve the perturbative expansion using the formulation advanced in [27].

To achieve this purpose we introduce a renormalization constant for the occupation number that absorbs the secular divergences at a fixed time scale τ and write

$$\delta n_k(t_o) = \delta n_k(\tau)\ \mathcal{Z}(\tau, t_o)\ \ ;\ \ \mathcal{Z}(\tau, t_o) = 1 + \alpha\ z_1(\tau, t_o) + \cdots \tag{3.5}$$

and request that the coefficients z_n cancel the secular divergences proportional to α^n at a given time scale τ. To lowest order the choice

$$z_1(\tau, t_o) = \int_{t_o}^{\tau} \Gamma_k(t')\ dt' \tag{3.6}$$

leads to the renormalized distribution function at time t in terms of the updated occupation number at the time scale τ

$$\delta n_k(t) = \delta n_k(\tau) \left\{ 1 - \alpha \int_{\tau}^{t} \Gamma_k(t')\ dt' \right\}$$

However, the occupation number $\delta n_k(t)$ cannot depend on the arbitrary renormalization scale τ, this independence on the renormalization scale leads to the renormalization group equation to lowest order:

$$\frac{\partial \delta n_k(\tau)}{\partial \tau} + \alpha\ \Gamma_k(\tau)\ \delta n_k(\tau) = 0\ . \tag{3.7}$$

This renormalization group equation is now clearly of the form of a Boltzmann equation in the relaxation time approximation with a time dependent rate.

Now choosing the renormalization scale to coincide with the time t in the solution of (3.7) as is usually done in the scaling analysis of the solutions to the renormalization group equations, we find that the distribution function in the linearized approximation evolves in time in the following manner

$$\delta n_k(t) = \delta n_k(t_o)\ e^{-\alpha \int_{t_o}^{t} \Gamma_k(t')\ dt'} \tag{3.8}$$

with the initial conditions

$$\delta n_k(t = t_o) = \delta n_k(t_o)\ ;\ \ \delta \dot{n}_k(t)\,|_{t=t_o}\ = 0 \tag{3.9}$$

in agreement with the perturbative expression (3.1). In the long time limit $\overline{\mu}(t - t_o) >> 1$ and using (3.4) we find that the distribution function relaxes towards equilibrium as

$$\delta n_k(t) \overset{\mu(t-t_o)>>1}{=} \delta n_k(t_o) e^{-2\alpha T (t-t_o) \ln[(t-t_o)\bar{\mu}]} \tag{3.10}$$

The exponent in (3.10) must be compared to that found for the relaxation of the mean field $\langle \Phi(\vec{k}, t) \rangle$ in [27]. We thus find that the distribution function approaches equilibrium as the *square* of the mean field, a result which is in agreement with the usual relation between the damping and the interaction rate.

Furthermore, the dynamical renormalization group resummation of the perturbative expansion reveals a relaxation time scale for the distribution function of the hard charged particles given by

$$t_{rel} \approx |\alpha T \ln \alpha|^{-1}\ .$$

IV. INTERPRETATION OF THE RG IN KINETICS

The interpretation of eq.(3.4) and the renormalization of the distribution function given by eq.(3.5) are clear: having prepared the initial distribution at a time t_o we evolve the distribution forward in time using perturbation theory but during a time scale τ such that the perturbative expansion is still valid, i.e. $t_{rel} >> (\tau - t_o)$. Secular terms begin to dominate the perturbative expansion at a time scale $(\tau - t_o) >> \omega_p^{-1}$, if there is a separation of time scales such that $t_{rel} >> (\tau - t_o) >> \omega_p^{-1}$ perturbation theory is reliable in this regime but secular terms appear and can be isolated. Thus the perturbative expansion is valid for a large time scale, at a given *renormalization* scale τ within this interval the result of the perturbative expansion is to 'reset' the occupation number, from $\delta n_k(t_0)$ to $\delta n_k(\tau)$. Having 'reset' the occupation number at this time scale in which perturbation theory is reliable, this new occupation number is used as an initial condition to iterate forward in time using perturbation theory to another time that again is within the perturbatively reliable region.

The renormalization condition (3.5) is precisely the resetting of the occupation number. Now the perturbative expansion has been improved and can be extended to longer time scales by using the updated occupation number as an initial condition at the scale τ. The dynamical renormalization group equation (3.7) is the differential form of the procedure of 'resetting' followed by perturbative evolution in real time. This renormalization group equation (3.7) is identified with the Boltzmann equation in the relaxation time approximation [31].

We note that there is certain freedom in the choice of the 'renormalization counterterms' z_n (3.5): rather than choosing z_1 as in eq.(3.6) we could have chosen to cancel *only the secular part* given by (3.4) rather than the whole integral. In the language of renormalization this choice would correspond to choosing the counterterms to cancel only the *divergent* contribution, i.e. we could have chosen

$$z_1 = 2T(\tau - t_o) \ln[\bar{\mu}(\tau - t_o)] \tag{4.1}$$

which would lead to the renormalization group equation

$$\frac{\partial \delta n_k(\tau)}{\partial \tau} + 2\alpha T \; \{\ln[\bar{\mu}(\tau - t_o)] + 1\} \; \delta n_k(\tau) = 0 \tag{4.2}$$

Now choosing the renormalization scale τ with the time t in the solution of (4.2) we would find the asymptotic form

$$\delta n_k(t) = \delta n_k(t_o) \, e^{-2\alpha T(t-t_o)\ln[(t-t_o)\bar{\mu}]}$$

Obviously the two choices (3.6) and (4.1) lead to the same asymptotic behavior, because they only differ by non-secular terms which are subleading in the asymptotic regime. The choice (3.6) also contains non-secular contributions which are bound in time and therefore always perturbative, i.e. *finite* terms in the usual renormalization program. The advantage of choosing (3.6) is that the solution (3.8) includes transient effects and obeys the initial conditions (3.9) explicitly.

In this particular case, the validity of the renormalization group approach hinges upon a separation of time scales: the time scale at which the occupation number is reset, i.e. τ has to be close to the scale t so that the perturbative expansion has been improved but not too long so that the occupation number has changed much during this scale. Since the scale of the kernel is given by the plasma frequency $\omega_p \approx eT$ which enters in the self-energy of the HTL resummed photon and the infrared cutoff μ, the separation of scales is guaranteed in weak coupling since $t_{rel} \approx |\alpha T \ln(\alpha)|^{-1} >> \omega_p^{-1} \approx (eT)^{-1}$. This is a general statement for the validity of *any* kinetic approach, that is, there must be a clear separation of scales between the relaxation and the microscopic time scales.

This interpretation of kinetics and the Boltzmann equation via the dynamical renormalization group provides a natural answer to the question: What does the Boltzmann equation sum??. In a typical kinetic approach one writes a Boltzmann equation by computing in perturbation theory some transition matrix element which enters in an integral kernel. Obviously the full solution of the Boltzmann equation is *non-perturbative* even when the transition matrix element has been computed in perturbation theory. In particular in the relaxation time approximation the coupling constant appears in the solution in an exponential form via the relaxation time or damping rate. What our analysis in terms of the renormalization group reveals is that what is being summed by the Boltzmann equations are *secular terms in time*.

In this article we have focused on the derivation of the kinetic equation and the renormalization group resummation in the relaxation time approximation to compare the relaxation of the distribution function of hard charged scalars to that of the single particle Green's function [25,27]. We relegate the treatment of the full non-linear kinetic equations via the dynamical renormalization in its full generality to a forthcoming article [31].

V. CONCLUSIONS, IMPLICATIONS AND MORE QUESTIONS:

Our goal with this article was to understand the relaxation of the distribution function of charged fields with hard momenta in a hot gauge theory. The detailed study of the relaxation of the single charged (quasi)particle Green's function in the eikonal (Bloch-Nordsieck) approximation in references [25,26] revealed an anomalous relaxation as a consequence of the infrared divergences associated with the emission and absorption of HTL resummed transverse photons. This novel form of relaxation of the single particle Green's function was confirmed in terms of a dynamical renormalization group resummation in [27].

There are several important implications of these results:

- Although in QED there is no magnetic mass, the analisis of [25–27] reveals that the time scale for relaxation of fast moving single particle charged excitations is given by $2|\alpha T \ln(\alpha)|^{-1}$. This is the *same* time scale that arises in QCD after resummation and assuming that a magnetic mass $m_{mag} << m_D \approx gT$ cuts off the infrared divergences in the damping rate [21,22]. Thus the conjecture in [21] is confirmed through a non-exponential relaxation. The advantage of a real time analisis is that the time variable

serves as an infrared cutoff. Furthermore the renormalization group approach is not restricted to the quasiparticle picture.

- The *distribution function* of fast charged excitations in the linearized approximation relaxes also with an an anomalous, non-exponential form that asymptotically is the *square* of the single particle Green's function. Therefore this provides another confirmation of the relation between the 'damping' and 'interaction' rates but with non-exponential relaxation. The relaxation time scale for the distribution function is given by $t_{rel} \approx |\alpha T \ln(\alpha)|^{-1}$, i.e. one half of the relaxation time scale for the single particle Green's function.

- This result suggests that at very large energy when the QCD coupling is small, i.e. in the initial stages of an Ultrarelativistic Heavy Ion Collision, the thermalization rate of quarks could be as fast as that of gluons. The in-medium effects that lead to the logarithmic enhancement of the thermalization scale could compensate for the group factors that result in a larger thermalization rate for gluons.

There are still unresolved and important questions that deserve attention. In particular the treatment of collective modes. Here we have focused on hard excitations, and in the limit of large momenta $k >> eT$ collective and single particle excitations are indistinguishable. Hence the definition of the number of charged excitations coincides with that for asymptotic single particle states. However for $k \leq eT$ we must treat the distribution function of the collective modes in terms of an interpolating operator that counts these collective modes. Furthermore for soft collective modes, not only the internal photon but the charged particle propagator and the vertices must be HTL resummed [21,25] and this situation must be studied in detail. We hope to report on progress on some of these issues along with a treatment of non-linear kinetics and resolution of pinch singularities with the dynamical renormalization group in a forthcoming article [31].

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